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Characterization of dependence of multidimensional Lévy processes using Lévy copulas ☆

Jan Kallsen[a], Peter Tankov[b, c, *]

[a] HVB-Stiftungsinstitut für Finanzmathematik, Zentrum Mathematik, TU München, Boltzmannstraße 3, 85747 Garching bei München, Germany
[b] INRIA Rocquencourt—B.P. 105, 78153 Le Chesnay Cedex, France
[c] Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VII, 2, Place Jussieu, 75251 Paris Cedex 05, France

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Abstract

This paper suggests *Lévy copulas* in order to characterize the dependence among components of multidimensional Lévy processes. This concept parallels the notion of a copula on the level of Lévy measures. As for random vectors, a version of Sklar's theorem states that the law of a general multivariate Lévy process is obtained by combining arbitrary univariate Lévy processes with an arbitrary Lévy copula. We construct parametric families of Lévy copulas and prove a limit theorem, which indicates how to obtain the Lévy copula of a multivariate Lévy process X from the ordinary copula of the random vector X_t for small t.

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1. Introduction

Copulas allow to separate the dependence structure of a random vector from its univariate margins. Their role is twofold. Firstly, they provide a complete characterization of the possible dependence structures of a random vector with fixed margins. Secondly, they can be used to

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* Corresponding author.
E-mail addresses: kallsen@ma.tum.de (J. Kallsen), Peter.Tankov@polytechnique.org (P. Tankov).

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construct multidimensional distributions with specified dependence and arbitrary marginal laws. Despite the presence of a vast body of literature on copulas and, more generally, on the dependence of random vectors [6,14,20], few efforts have been made to study dependence in the dynamic context of stochastic processes. This paper aims to partially fill this gap by addressing the dependence among the components of multivariate Lévy processes.

The first goal of this study is to characterize all $\mathbb{R}^d$-valued Lévy processes X whose components $X^1, \dots, X^d$ are equal in law to d given univariate Lévy processes $Y^1, \dots, Y^d$, respectively. In principle, the whole distribution of a d-dimensional Lévy process $X = (X_t)_{t \in \mathbb{R}_+}$ is determined by the law of X_t for one fixed t. Therefore, one could describe the dependence among the components of X by the copula C_t of X_t. However, two problems arise:

- For given infinitely divisible univariate laws μ, ν it is unclear which copulas yield a two-dimensional infinitely divisible law. Moreover, the answer to this question depends strongly on μ and ν. The class $\mathcal{C}_{\text{id}}$ of copulas that yield two-dimensional infinitely divisible distributions for *all* infinitely divisible marginal laws is almost empty. Indeed, consider the case when μ and ν are Gaussian. Then the two-dimensional distribution is infinitely divisible if and only if it is Gaussian as well. Hence $\mathcal{C}_{\text{id}}$ cannot contain copulas other than Gaussian. Now suppose that μ, ν are Poisson with parameter 1. One can verify that the one-parametric family of infinitely divisible distributions having these marginal laws does not have a Gaussian copula unless we are in the independent or completely dependent case.
- The copula C_t of X_t typically depends on t (cf. [22] for an example). In general, C_s cannot be computed from C_t alone if $s \neq t$. One also needs to know the marginal distributions at time t and at time s. It is not possible to choose the family $(C_t)_{t \in \mathbb{R}_+}$ of copulas and the marginal laws of $(X_t)_{t \in \mathbb{R}_+}$ independently. If one changes the margins, the family of copulas changes as well. On the other hand, there are very few examples where C_s can be calculated explicitly from C_t and the margins of the Lévy process. Numerical computation of this quantity is very demanding since it requires two multivariate Fourier transforms.

As a consequence and since most Lévy models in the literature specify the Lévy–Khintchine triplet (a, ν, γ) of the process, it is more satisfactory to use the latter in order to characterize the dependence structure of a Lévy process in a time-independent fashion. The location parameter γ does not play a role in this context. The dependence structure of the Brownian motion part of the Lévy process is characterized entirely by its covariance matrix a. Since the continuous part and the jump part of X are stochastically independent, it remains to describe the dependence structure of the jump part of X.

Starting from the historical monograph by Lévy [9], many authors have analyzed the structure of Lévy measures of different subclasses of multivariate Lévy processes. For the most studied class of α-stable Lévy processes [18], the Lévy measure ν satisfies

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}} \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a finite measure on the unit sphere S. Rosiński [16] defines the class of multivariate tempered stable processes which combine some properties of both α-stable processes and Brownian motions. Their Lévy measure is given by

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{q(r, \xi)\, dr}{r^{1+\alpha}} \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d), \tag{1.1}$$

where $q : (0, \infty) \times S \to (0, \infty)$ is a Borel function such that $q(\cdot, \xi)$ is completely monotone with $q(0+, \xi) = 1$ and $q(\infty, \xi) = 0$ for any $\xi \in S$.

Multivariate Lévy processes with dependent components can also be obtained by subordination, either in the classical sense by time-changing a multivariate Lévy process with a one-dimensional increasing Lévy process [19,15], or in the sense of multivariate subordination as in [2]. When the subordinated process is a Brownian motion, both procedures yield Lévy processes whose distributions at any given time are of so-called type G. Multivariate infinitely divisible distributions of type G are studied in [12] and [13]. Their Lévy measures admit a representation similar to (1.1).

In this paper, we study the structure of general Lévy measures instead of concentrating on a particular subclass of processes. We show that any such Lévy measure can be constructed from the marginal Lévy measures and a new object, the *Lévy copula*, which describes the dependence between components and does not depend on their individual laws. In the particular case of two-dimensional processes whose components have atomless Lévy measures and admit only positive jumps, this concept is discussed in [3] but it is introduced here for the first time in the context of general Lévy processes. A version of Sklar's theorem states that, as for random vectors, the margins and the dependence structure of a Lévy process can be modelled independently (cf. Theorem 3.6). This suggests to construct multidimensional Lévy processes by combining arbitrary one-dimensional Lévy processes with a Lévy copula from a parametric family (cf. Section 6).

The second aim of this work is to express special dependence structures of Lévy processes such as complete dependence and independence in terms of Lévy copulas. This is done in Section 4, where we also characterize the dependence structure of stable Lévy motions in terms of Lévy copulas.

The Lévy copula and the Gaussian copula of the Brownian motion part of a Lévy process X can be recovered from the ordinary copula of X_t at small fixed times t. This relation is established in Section 5 by way of limit theorems.

Our final objective is to construct parametric families of Lévy copulas which may turn out to be useful in applications. In Section 6, we discuss a possible approach which allows to build families of Lévy copulas in arbitrary dimension where the number of parameters does not depend on the dimension. This is motivated by the observation that typically one does not have enough information about the dependence structure to estimate many parameters or to proceed with a nonparametric approach.

An important field of application of Lévy copulas is mathematical finance. Many problems in this domain require a multivariate model with dependence between components, where jumps in assets are taken into account. While Lévy processes with jumps have been successfully applied by many authors to construct one-dimensional models (cf. e.g. [1,4,8,11,17]), multivariate applications continue to be dominated by Brownian motion. [3] discusses examples of multidimensional models with jumps, which are constructed with the help of Lévy copulas.

2. Preliminaries

In this section, we recall a few facts on increasing functions. We set $\overline{\mathbb{R}} := (-\infty, \infty]$ in this paper and

$$\operatorname{sgn} x := \begin{cases} 1 & \text{for } x \geqslant 0, \\ -1 & \text{for } x < 0. \end{cases}$$

For $a, b \in \overline{\mathbb{R}}^d$ we write $a \leqslant b$ if $a_k \leqslant b_k$, $k = 1, \ldots, d$. In this case, let $(a, b]$ denote a right-closed left-open interval of $\overline{\mathbb{R}}^d$:

$$(a, b] := (a_1, b_1] \times \cdots \times (a_d, b_d].$$

Definition 2.1. Let $F : S \to \overline{\mathbb{R}}$ for some subset $S \subset \overline{\mathbb{R}}^d$. For $a, b \in S$ with $a \leqslant b$ and $\overline{(a,b]} \subset S$, the *F-volume* of $(a, b]$ is defined by

$$V_F((a,b]) := \sum_{u \in \{a_1,b_1\} \times \cdots \times \{a_d,b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) := \#\{k : u_k = a_k\}$.

In particular, $V_F((a, b]) = F(b) - F(a)$ for $d = 1$ and $V_F((a, b]) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$ for $d = 2$. If $F(u) = \prod_{i=1}^d u_i$, the F-volume of any interval is equal to its Lebesgue measure.

Definition 2.2. A function $F : S \to \overline{\mathbb{R}}$ for some subset $S \subset \overline{\mathbb{R}}^d$ is called *d-increasing* if $V_F((a, b]) \geqslant 0$ for all $a, b \in S$ with $a \leqslant b$ and $\overline{(a,b]} \subset S$.

Example 2.3. The distribution function F of a random vector $X \in \mathbb{R}^d$ is usually defined by

$$F(x_1, \ldots, x_d) := P[X_1 \leqslant x_1, \ldots, X_d \leqslant x_d]$$

for $x_1, \ldots, x_d \in \overline{\mathbb{R}}$. F is then clearly increasing because

$$V_F((a,b]) = P[X \in (a,b]] \tag{2.1}$$

for every $a, b \in \overline{\mathbb{R}}^d$ with $a \leqslant b$.

Definition 2.4. Let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a d-increasing function such that $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$. For any non-empty index set $I \subset \{1, \ldots, d\}$, the *I-margin* of F is the function $F^I : \overline{\mathbb{R}}^{|I|} \to \overline{\mathbb{R}}$, defined by

$$F^I((u_i)_{i \in I}) := \lim_{a \to \infty} \sum_{(u_i)_{i \in I^c} \in \{-a, \infty\}^{|I^c|}} F(u_1, \ldots, u_d) \prod_{i \in I^c} \operatorname{sgn} u_i,$$

where $I^c := \{1, \ldots, d\} \setminus I$.

In particular, we have $F^{\{1\}}(u) = F(u, \infty) - \lim_{a \to -\infty} F(u, a)$ for $d = 2$. To understand the reasoning leading to the above definition of margins, note that any positive measure μ on $\overline{\mathbb{R}}^d$ naturally induces an increasing function F via

$$F(u_1, \ldots, u_d) := \mu((u_1 \wedge 0, u_1 \vee 0] \times \cdots \times (u_d \wedge 0, u_d \vee 0]) \prod_{i=1}^d \operatorname{sgn} u_i \tag{2.2}$$

for $u_1, \ldots u_d \in \overline{\mathbb{R}}$. The margins of μ are usually defined by

$$\mu^I(A) = \mu\left(\{u \in \overline{\mathbb{R}}^d : (u_i)_{i \in I} \in A\}\right), \quad A \subset \overline{\mathbb{R}}^{|I|} \tag{2.3}$$

It is now easy to see that the margins of F are induced by the margins of μ in the sense of (2.2).

3. Definition of Lévy copulas

As it is explained in the introduction, the dependence structure of a multivariate Lévy process can be reduced to the Lévy measure and the covariance matrix of the Gaussian part. Since the

Lévy measure is a measure on $\mathbb{R}^d$, it is possible to define a suitable notion of a copula. However, one has to take care of the fact that the Lévy measure is possibly infinite with a singularity at the origin.

Definition 3.1. A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called *Lévy copula* if
(1) $F(u_1, \dots, u_d) \neq \infty$ for $(u_1, \dots, u_d) \neq (\infty, \dots, \infty)$,
(2) $F(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$,
(3) F is d-increasing,
(4) $F^{\{i\}}(u) = u$ for any $i \in \{1, \dots, d\}$, $u \in \mathbb{R}$.

The next lemma establishes that, similarly to ordinary copulas [14, Th. 2.10.7], Lévy copulas are Lipschitz continuous.

Lemma 3.2. *Let F be a Lévy copula and $u, v \in \mathbb{R}^d$. Then*

$$|F(v_1, \dots, v_d) - F(u_1, \dots, u_d)| \leqslant \sum_{i=1}^{d} |v_i - u_i|.$$

Proof. It is easy to see that it suffices to consider the case $u_i v_i \geqslant 0$ for $i = 1, \dots, d$. To simplify notation, we suppose that $0 \leqslant u_i \leqslant v_i$ for every i. The general case can be treated similarly.

$$\begin{aligned}
&|F(v_1, \dots, v_d) - F(u_1, \dots, u_d)| \\
&\quad = |V_F((0, v_1] \times \cdots \times (0, v_d]) - V_F((0, u_1] \times \cdots \times (0, u_d])| \\
&\quad \leqslant \sum_{i=1}^{d} \lim_{a \to \infty} V_F\left((-a, \infty]^{i-1} \times (u_i, v_i] \times (-a, \infty]^{d-i}\right) \\
&\quad = \sum_{i=1}^{d} \left(F^{\{i\}}(v_i) - F^{\{i\}}(u_i)\right) \\
&\quad = \sum_{i=1}^{d} |v_i - u_i|
\end{aligned}$$

as claimed. □

In the sequel, we will need a special interval associated with any $x \in \mathbb{R}$:

$$\mathcal{I}(x) := \begin{cases} (x, \infty), & x \geqslant 0, \\ (-\infty, x], & x < 0. \end{cases}$$

In the same way as the distribution of a random vector can be represented by its distribution function, the Lévy measure of a Lévy process will be represented by its tail integral.

Definition 3.3. Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν. The *tail integral* of X is the function $U : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}$ defined by

$$U(x_1, \dots, x_d) := \prod_{i=1}^{d} \operatorname{sgn}(x_i) \nu \left(\prod_{j=1}^{d} \mathcal{I}(x_j) \right). \tag{3.1}$$

In principle, the tail integral could be defined on $\mathbb{R}^d$ instead of $(\mathbb{R} \setminus \{0\})^d$ using (3.1), but with this definition the main representation formula (3.2) does not hold in its present form. The tail integral in Definition 3.3 does not determine the Lévy measure uniquely because it does not reflect the mass on the coordinate axes. E.g. the tail integral equals 0 for a Lévy process with independent components. However, we will see that the Lévy measure is completely determined by its tail integral and all its marginal tail integrals.

Definition 3.4. Let X be a $\mathbb{R}^d$-valued Lévy process and let $I \subset \{1, \dots, d\}$ be non-empty. The *I-marginal tail integral* U^I of X is the tail integral of the process $X^I := (X^i)_{i \in I}$. To simplify notation, we denote one-dimensional margins by $U_i := U^{\{i\}}$.

Lemma 3.5. *Let X be a* $\mathbb{R}^d$*-valued Lévy process. Its marginal tail integrals* $\{U^I : I \subset \{1, \dots, d\}$ non-empty$\}$ *are uniquely determined by its Lévy measure* ν. *Conversely, its Lévy measure is uniquely determined by the set of its marginal tail integrals.*

Proof. $\Rightarrow$: By Proposition 11.10 in [19], the Lévy measure of X^I is given by

$$\nu^I(A) = \nu(\{x \in \mathbb{R}^d : (x_i)_{i \in I} \in A \setminus \{0\}\}), \quad A \in \mathcal{B}(\mathbb{R}^{|I|})$$

for any non-empty $I \subset \{1, \dots, d\}$. Moreover, the marginal tail integrals are uniquely determined by the marginal Lévy measures.

$\Leftarrow$: It is sufficient to prove that $\nu((a, b])$ is completely determined by the tail integrals for any $a, b \in \mathbb{R}^d$ with $a \leqslant b$ and $0 \notin (a, b]$. We prove by induction on $k = 0, \dots, d$ that $\nu^I(\prod_{i \in I}(a_i, b_i])$ is determined by the tail integrals for any $a, b \in \mathbb{R}^d$ such that $a \leqslant b$ and $a_i b_i \leqslant 0$ for at most k indices and any non-empty $I \subset \{1, \dots, d\}$ with $0 \notin \prod_{i \in I}(a_i, b_i]$.

If $k = 0$, Definitions 3.3 and 3.4 entail that

$$\nu^I\left(\prod_{i \in I}(a_i, b_i]\right) = (-1)^{|I|} V_{U^I}\left(\prod_{i \in I}(a_i, b_i]\right).$$

Let $a, b \in \mathbb{R}^d$ such that $a_i b_i \leqslant 0$ for at most k indices. For ease of notation we suppose that $a_i b_i \leqslant 0$ for $i = 1, \dots, k$. Let $I \subset \{1, \dots, d\}$ non-empty with $0 \notin \prod_{i \in I}(a_i, b_i]$. By induction hypothesis, $\nu^I(\prod_{i \in I}(a_i, b_i])$ is uniquely determined if $k \notin I$. Suppose that $k \in I$. If $a_k = 0$, then

$$\nu^I\left(\prod_{i \in I}(a_i, b_i]\right) = \lim_{\alpha \downarrow 0} \nu^I\left(\prod_{i \in I, i < k}(a_i, b_i] \times (\alpha, b_k] \times \prod_{i \in I, i > k}(a_i, b_i]\right)$$

and the right-hand side is uniquely determined by the induction hypothesis. If $a_k \neq 0$, then

$$\begin{aligned}\nu^I\left(\prod_{i \in I}(a_i, b_i]\right) = \nu^{I \setminus \{k\}}&\left(\prod_{i \in I \setminus \{k\}}(a_i, b_i]\right) \\ &- \lim_{\beta \downarrow b_k; c \uparrow \infty} \nu^I\left(\prod_{i \in I, i < k}(a_i, b_i] \times (\beta, c] \times \prod_{i \in I, i > k}(a_i, b_i]\right) \\ &- \lim_{c \downarrow -\infty} \nu^I\left(\prod_{i \in I, i < k}(a_i, b_i] \times (c, a_k] \times \prod_{i \in I, i > k}(a_i, b_i]\right),\end{aligned}$$

which is uniquely determined as well. $\square$

The following theorem is our first main result. It explains the relation between Lévy copulas and Lévy processes. It may be called *Sklar's theorem for Lévy copulas*.

Theorem 3.6.(1) *Let $X = (X^1, \ldots, X^d)$ be a $\mathbb{R}^d$-valued Lévy process. Then there exists a Lévy copula F such that the tail integrals of X satisfy*

$$U^I((x_i)_{i\in I}) = F^I((U_i(x_i))_{i\in I}) \tag{3.2}$$

for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i\in I} \in (\mathbb{R}\setminus\{0\})^{|I|}$. The Lévy copula F is unique on $\prod_{i=1}^d \overline{\operatorname{Ran} U_i}$.

(2) *Let F be a d-dimensional Lévy copula and $U_i, i = 1, \ldots, d$ tail integrals of real-valued Lévy processes. Then there exists a $\mathbb{R}^d$-valued Lévy process X whose components have tail integrals $U_1, \ldots, U_d$ and whose marginal tail integrals satisfy Eq.* (3.2) *for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i\in I} \in (\mathbb{R}\setminus\{0\})^{|I|}$. The Lévy measure ν of X is uniquely determined by F and $U_i, i = 1, \ldots, d$.*

Remark. The main difficulty in proving this theorem is to construct a Lévy copula F from a given Lévy measure. This becomes almost trivial if the one-dimensional marginal Lévy measures are infinite and have no atoms: In this case $\operatorname{Ran} U_i = (-\infty, 0) \cup (0, \infty)$ for any i and one can construct F directly via

$$F(u_1, \ldots, u_d) = U\left(U_1^{-1}(u_1), \ldots, U_d^{-1}(u_d)\right). \tag{3.3}$$

In the general case the idea is to construct a measure $\widetilde{m}$ which defines F via (2.2) for $\mu = \widetilde{m}$. This in turn is done by some kind of randomization of the marginal tail integrals at their points of discontinuity (cf. the first term in (3.4)) and at zero (cf. the second term in (3.4)).

Proof. 1. Denote the Lévy measure of X and $X^1, \ldots, X^d$ by ν and $\nu_1, \ldots, \nu_d$, respectively. For the purposes of this proof we set for $x \in \overline{\mathbb{R}}$, $i = 1, \ldots, d$,

$$\dot{U}_i(x) := \begin{cases} U_i(x) & \text{for } x \neq 0 \text{ and } x \neq \infty, \\ 0 & \text{for } x = \infty, \\ \infty & \text{for } x = 0 \end{cases}$$

and

$$\Delta U_i(x) := \begin{cases} \lim\limits_{\xi\uparrow x} U_i(\xi) - U_i(x) & \text{for } x \neq 0 \text{ and } x \neq \infty, \\ 0 & \text{for } x = \infty \text{ or } x = 0. \end{cases}$$

The construction of F via (3.3) does not work if $\nu_1, \ldots, \nu_d$ have atoms or if they are finite. The way out is to replace ν by an atomless infinite measure on some larger space. This measure m on $(\overline{\mathbb{R}}^d \setminus \{0\}) \times [0, 1]^d \times \mathbb{R}$ is defined by

$$\begin{aligned} m &:= \nu^* \otimes \lambda|_{(0,1)^d} \otimes \varepsilon_0 \\ &\quad + \sum_{i=1}^d \varepsilon_{(\underbrace{0,\ldots,0}_{i-1},\infty,0,\ldots,0)} \otimes \varepsilon_{(\underbrace{0,\ldots,0}_{d})} \otimes \lambda|_{(\nu_i((0,\infty)),\infty)\cup(-\infty,-\nu_i((-\infty,0)))}, \end{aligned} \tag{3.4}$$

where ν^* is the extension of ν to $\overline{\mathbb{R}}^d \setminus \{0\}$, i.e. $\nu^*(A) := \nu(A \cap \mathbb{R}^d)$. Let

$$g_i : \overline{\mathbb{R}} \times [0, 1] \times \mathbb{R} \to \overline{\mathbb{R}}, \quad (x, y, z) \mapsto \dot{U}_i(x) + y\Delta U_i(x) + z$$

and define a measure $\widetilde{m}$ on $\overline{\mathbb{R}}^d \setminus \{\infty, \ldots, \infty\}$ via

$$\widetilde{m}(B) := m(\widetilde{g}^{-1}(B)) \tag{3.5}$$

with

$$\widetilde{g}(x_1, \ldots, x_d, y_1, \ldots, y_d, z) := (g_1(x_1, y_1, z), \ldots, g_d(x_d, y_d, z)).$$

Eq. (3.5) plays the role of (3.3) on the level of measures rather than tail integrals. Finally, let F be given by

$$F(u_1, \ldots, u_d) := \widetilde{m}\left(\prod_{i=1}^{d} (u_i \wedge 0, u_i \vee 0]\right) \prod_{i=1}^{d} \operatorname{sgn} u_i$$

for $(u_1, \ldots, u_d) \in \overline{\mathbb{R}}^d$. Properties (1) and (2) in Definition 3.1 are obvious. From the fact that $\widetilde{m}$ is a positive measure it follows immediately that F is d-increasing. Let $I \subset \{1, \ldots, d\}$, $(u_i)_{i\in I} \in \overline{\mathbb{R}}^{|I|}$. For ease of notation, we consider only the case of non-negative u_i. The general case follows analogously. By definition of F we have

$$\begin{aligned}
F^I((u_i)_{i\in I}) &= \lim_{a\to\infty} \sum_{(u_j)_{j\in I^c} \in \{-a,\infty\}^{|I^c|}} F(u_1, \ldots, u_d) \prod_{j\in I^c} \operatorname{sgn} u_j \\
&= \widetilde{m}\left(\prod_{i\in I} (0, u_i] \times \overline{\mathbb{R}}^{|I^c|}\right) \\
&= m\left(\Big\{(x_1, \ldots, x_d, y_1, \ldots, y_d, z) \in (\overline{\mathbb{R}}^d \setminus \{0\}) \times [0,1]^d \times \mathbb{R} : \right. \\
&\qquad \left. \dot{U}_i(x_i) + y_i \Delta U_i(x_i) + z \in (0, u_i] \text{ for } i \in I\Big\}\right).
\end{aligned}$$

If $I = \{i\}$, then the definition of m implies that this equals

$$\begin{aligned}
&\left(\nu_i \otimes \lambda|_{(0,1)}\right)\left(\{(x, y) \in \mathbb{R} \times [0,1] : \dot{U}_i(x) + y\Delta U_i(x) \in (0, u_i]\}\right) \\
&\quad + (u_i - \nu_i((0,\infty)))1_{\{u_i > \nu_i((0,\infty))\}}.
\end{aligned}$$

Introducing $x^* := \inf\{x \geqslant 0 \ : \ \dot{U}_i(x) + \Delta U_i(x) \leqslant u_i\}$, this can be expressed as

$$\nu_i((x^*, \infty)) + (u_i - \dot{U}_i(x^*))1_{\{x^* \neq 0\}} + (u_i - \nu_i((0,\infty)))1_{\{x^*=0\}} = u_i,$$

i.e. property (4) in Definition 3.1 is met.

Now, let $(x_i)_{i\in I} \in (\mathbb{R}\setminus\{0\})^{|I|}$. Again, we consider only the case where all the x_i are nonnegative. Then

$$\begin{aligned}
F^I((U_i(x_i))_{i\in I}) &= m\left(\Big\{(\widetilde{x}_1, \ldots, \widetilde{x}_d, y_1, \ldots, y_d, z) \in \overline{\mathbb{R}}^d \times [0,1]^d \times \mathbb{R} : \right. \\
&\qquad \left. \dot{U}_i(\widetilde{x}_i) + y_i \Delta U_i(\widetilde{x}_i) + z \in (0, U_i(x_i)] \text{ for } i \in I\Big\}\right) \\
&= \nu\left(\prod_{i\in I} (x_i, \infty) \times \mathbb{R}^{|I^c|}\right) \\
&= \nu^I\left(\prod_{i\in I} (x_i, \infty)\right) \\
&= U^I((x_i)_{i\in I})
\end{aligned}$$

as claimed. The uniqueness statement follows from (3.2) and Lemma 3.2.

2. Since F is d-increasing and continuous (by Lemma 3.2), there exists a unique measure μ on $\overline{\mathbb{R}}^d \setminus \{\infty, \dots, \infty\}$ such that $V_F((a, b]) = \mu((a, b])$ for any $a, b \in \overline{\mathbb{R}}^d \setminus \{\infty, \dots, \infty\}$ with $a \leqslant b$ (see [7], Section 4.5). For a one-dimensional tail integral $U(x)$, we define

$$U^{-1}(u) = \begin{cases} \inf\{x > 0 : u \geqslant U(x)\}, & u \geqslant 0, \\ \inf\{x < 0 : u \geqslant U(x)\} \wedge 0, & u < 0. \end{cases}$$

Let $\nu' := f(\mu)$ be the image of μ under

$$f : (u_1, \dots, u_d) \mapsto (U_1^{-1}(u_1), \dots, U_d^{-1}(u_d))$$

and let ν be the restriction of ν' to $\mathbb{R}^d \setminus \{0\}$. We need to prove that ν is a Lévy measure and that its marginal tail integrals U_ν^I satisfy

$$U_\nu^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I})$$

for any non-empty $I \subset \{1, \dots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. Suppose for ease of notation that $x_i > 0$, $i \in I$. Then

$$\begin{aligned} U_\nu^I((x_i)_{i \in I}) &= \nu(\{\xi \in \mathbb{R}^d : \xi_i \in (x_i, \infty), i \in I\}) \\ &= \mu(\{u \in \overline{\mathbb{R}}^d : U_i^{-1}(u_i) \in (x_i, \infty), i \in I\}) \\ &= \mu(\{u \in \overline{\mathbb{R}}^d : 0 < u_i < U(x_i), i \in I\}). \end{aligned}$$

By Lemma 3.2 we have $\mu(\{u \in \overline{\mathbb{R}}^d : u_i = U(x_i)\}) = 0$ for $i \in I$. Therefore,

$$\begin{aligned} U_\nu^I((x_i)_{i \in I}) &= \mu(\{u \in \overline{\mathbb{R}}^d : 0 < u_i \leqslant U(x_i), i \in I\}) \\ &= F^I((U_i(x_i))_{i \in I}). \end{aligned}$$

This proves in particular that the one-dimensional marginal tail integrals of ν equal $U_1, \dots, U_d$.

Since the marginals ν_i of ν are Lévy measures on $\mathbb{R}$, we have $\int (x_i^2 \wedge 1)\nu_i(dx_i) < \infty$ for $i = 1, \dots, d$. This implies

$$\begin{aligned} \int (|x|^2 \wedge 1)\nu(dx) &\leqslant \int \sum_{i=1}^d (x_i^2 \wedge 1)\nu(dx) \\ &= \sum_{i=1}^d \int (x_i^2 \wedge 1)\nu_i(dx_i) < \infty \end{aligned}$$

and hence ν is a Lévy measure on $\mathbb{R}^d$. The uniqueness of ν follows from the fact that it is uniquely determined by its marginal tail integrals (cf. Lemma 3.5). □

Definition 3.7. We call any Lévy copula as given in part (1) of Theorem 3.6 a *Lévy copula of the Lévy process X*.

Lévy copulas are not limited to Lévy processes. A large class of Markov processes or even semimartingales behaves locally as a Lévy process in the sense that its dynamics can be described by a drift rate, a covariance matrix, and a Lévy measure, which may all change randomly through time [5, II.2.9, II.4.19]. Therefore, the notion of Lévy copula could naturally be extended to these more general classes of processes.

4. Examples of Lévy copulas

We start by characterizing independence of the components of a Lévy process in terms of its Lévy copula.

Proposition 4.1. *The components $X^1, \dots, X^d$ of a $\mathbb{R}^d$-valued Lévy process X are independent if and only if their Brownian motion parts are independent and if X has a Lévy copula of the form*

$$F_\perp(x_1, \dots, x_d) := \sum_{i=1}^d x_i \prod_{j \neq i} 1_{\{\infty\}}(x_j). \tag{4.1}$$

Proof. It is straightforward to see that Eq. (4.1) defines a Lévy copula.

$\Leftarrow$: Let $I \subset \{1, \dots, d\}$ with card $I \geqslant 2$. Definition 2.4 entails that $F_\perp^I((u_i)_{i\in I}) = 0$ for all $(u_i)_{i\in I} \in \mathbb{R}^{|I|}$. Therefore, $U^I((x_i)_{i\in I}) = 0$ for all $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$ by (3.2), which implies that the Lévy measure of X is supported by the union of the coordinate axes. A result in [19] (E 12.10 on p. 67) now allows to conclude that $X^1, \dots, X^d$ are independent.

$\Rightarrow$: Independence of $X^1, \dots, X^d$ implies that the Lévy measure of X is supported by the union of the coordinate axes (E 12.10 on p. 67 in [19]). Therefore, $U^I((x_i)_{i\in I}) = 0$ for all $I \subset \{1, \dots, d\}$ with card $I \geqslant 2$ and all $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. Since the copula (4.1) satisfies $F_\perp^I((U_i(x_i))_{i\in I}) = 0$ for all $I \subset \{1, \dots, d\}$ with card $I \geqslant 2$ and all $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$, this implies that $F_\perp$ is a Lévy copula for X. $\square$

For the characterization of complete dependence we need a number of definitions. First of all recall that a subset S of $\mathbb{R}^d$ is called *ordered* if, for any two vectors $u, v \in S$, either $u_k \leqslant v_k$, $k = 1, \dots, d$ or $u_k \geqslant v_k$, $k = 1, \dots, d$. Similarly, S is called *strictly ordered* if, for any two different vectors $u, v \in S$, either $u_k < v_k$, $k = 1, \dots, d$ or $u_k > v_k$, $k = 1, \dots, d$. In the following definition and below we set

$$K := \{x \in \mathbb{R}^d : \operatorname{sgn} x_1 = \cdots = \operatorname{sgn} x_d\}. \tag{4.2}$$

Definition 4.2. Let X be a $\mathbb{R}^d$-valued Lévy process. Its jumps are said to be *completely dependent* or *comonotonic* if there exists a strictly ordered subset $S \subset K$ such that $\Delta X_t := X_t - X_{t-} \in S$, $t \in \mathbb{R}_+$ (except for some null set of paths).

Clearly, an element of a strictly ordered set is completely determined by one coordinate only. Therefore, if the jumps of a Lévy process are completely dependent, the jumps of all components can be determined from the jumps of any single component. If the Lévy process in question has no continuous martingale part, then the trajectories of all components can be determined from the trajectory of any component, which indicates that Definition 4.2 is a reasonable dynamic notion of complete dependence for Lévy processes. The condition $\Delta X_t \in K$ means that if the components of a Lévy process are comonotonic, they always jump in the same direction.

For any $\mathbb{R}^d$-valued Lévy process X with Lévy measure ν and for any $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ the number of jumps in the time interval $[0, t]$ with sizes in B is a Poisson random variable with parameter $t\nu(B)$. Therefore, Definition 4.2 can be restated equivalently as follows:

Definition 4.3. Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν. Its jumps are said to be *completely dependent* or *comonotonic* if there exists a strictly ordered subset S of K such that $\nu(\mathbb{R}^d \setminus S) = 0$.

The following theorem characterizes complete jump dependence in terms of Lévy copulas.

Theorem 4.4. *Let X be a $\mathbb{R}^d$-valued Lévy process whose Lévy measure is supported by an ordered set $S \subset K$, where K is defined in* (4.2). *Then the* complete dependence Lévy copula *given by*

$$F_{\|}(x_1, \ldots, x_d) := \min(|x_1|, \ldots, |x_d|) 1_K(x_1, \ldots, x_d) \prod_{i=1}^{d} \operatorname{sgn} x_i \tag{4.3}$$

is a Lévy copula of X.

Conversely, if $F_{\|}$ is a Lévy copula of X, then the Lévy measure of X is supported by an ordered subset of K. If, in addition, the tail integrals U_i of X^i are continuous and satisfy $\lim_{x\to 0} U_i(x) = \infty$, $i = 1, \ldots, d$, *then $F_{\|}$ is the unique Lévy copula of X and the jumps of X are completely dependent.*

The proof is based on the following representation of an ordered set as a union of a strictly ordered set and countable many segments that are parallel to some coordinate axis.

Lemma 4.5. *Let $S \subset \mathbb{R}^d$ be ordered. It can be written as*

$$S = S^* \cup \bigcup_{n=1}^{\infty} S_n, \tag{4.4}$$

where $S^* \subset \mathbb{R}^d$ is strictly ordered and for every n, $S_n \subset \mathbb{R}^d$ and there exist $k(n)$ and $\xi(n)$ such that $x_{k(n)} = \xi(n)$ for all $x \in S_n$.

Proof. For the purposes of this proof we define the length of an ordered set S' by $|S'| := \sup_{a,b\in S'} \sum_{i=1}^{d}(a_i - b_i)$. Let

$$S(\xi, k) = \{x \in \mathbb{R}^d : x_k = \xi\} \cap S. \tag{4.5}$$

Firstly, we want to prove that there is at most a countable number of such segments with non-zero length. Fix $\varepsilon > 0$ and $k \in \{1, \ldots, d\}$. Consider N different segments $S_i = S(\xi_i, k)$, $i = 1, \ldots, N$ with length $\geqslant \varepsilon$. Since the S_i are different, the ξ_i must be different from each other as well and we can suppose without loss of generality that $\xi_i < \xi_{i+1}$ for all i. Then $\overline{x}_i \leqslant \underline{x}_{i+1}$ for all i, where $\overline{x}_i$ and $\underline{x}_i$ are the upper and the lower bounds of S_i. Since all S_i are subsets of S, which is an ordered set, this implies that $|\bigcup_{i=1}^{N} S_i| \geqslant N\varepsilon$. Therefore, for all $A > 0$ and for all $\varepsilon > 0$, the set $[-A, A]^d$ contains a finite number of segments of type (4.5) with length greater or equal to ε. This means that there is at most a countable number of segments of non-zero length, which we denote by S_n, $n \in \mathbb{N}$.

Now let $S^* = S \setminus \bigcup_{n=1}^{\infty} S_n$. S^* is ordered because it is a subset of S. Let $x, y \in S^*$. If $x_k = y_k$ for some k, then either x and y are the same or they are in some segment of type (4.5) hence not in S^*. Therefore, either $x_k < y_k$ for every k or $x_k > y_k$ for every k, which entails that S^* is strictly ordered and we have obtained the desired representation for S. □

Proof of Theorem 4.4. We start by proving that $F_{\|}$ is indeed a Lévy copula in the sense of Definition 3.1. Properties (1) and (2) in this definition are obvious. To show property (3), introduce a positive measure μ on $\overline{\mathbb{R}}^d$ by

$$\mu(B) = \lambda(\{x \in \mathbb{R} : (\underbrace{x, \ldots, x}_{d \text{ times}}) \in B\}), \quad B \in \mathcal{B}(\overline{\mathbb{R}}^d),$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Then

$$V_{F_\parallel}((a, b]) = \mu((a, b]) \quad \text{for any } a \leqslant b. \tag{4.6}$$

To see this suppose first that for all i, $a_i \leqslant 0 \leqslant b_i$. Then the sum in Definition 2.1 has only two terms and

$$V_{F_\parallel}((a, b]) = \min(|a_1|, \ldots, |a_d|) + \min(b_1, \ldots, b_d)$$

which is clearly equal to $\mu((a, b])$ (see the definition of μ). Now (4.6) can be shown by an induction argument: suppose that $a_i b_i > 0$ for at most k indices. Choose one such index and suppose for convenience that $0 < a_i < b_i$. Then

$$\begin{aligned} V_{F_\parallel}((a, b]) &= V_{F_\parallel}((a_1, b_1] \times \cdots \times (a_{i-1}, b_{i-1}] \times (0, b_i] \times (a_{i+1}, b_{i+1}] \times \cdots \times (a_d, b_d]) \\ &\quad - V_{F_\parallel}((a_1, b_1] \times \cdots \times (a_{i-1}, b_{i-1}] \times (0, a_i] \times (a_{i+1}, b_{i+1}] \times \cdots \times (a_d, b_d]) \\ &= \mu((a_1, b_1] \times \cdots \times (a_{i-1}, b_{i-1}] \times (0, b_i] \times (a_{i+1}, b_{i+1}] \times \cdots \times (a_d, b_d]) \\ &\quad - \mu((a_1, b_1] \times \cdots \times (a_{i-1}, b_{i-1}] \times (0, a_i] \times (a_{i+1}, b_{i+1}] \times \cdots \times (a_d, b_d]) \\ &= \mu((a, b]) \end{aligned}$$

by the induction hypothesis and Definition 2.1. Therefore, (4.6) is shown, which proves that $F_\parallel$ is d-increasing.

The margins of $F_\parallel$ have the same form as $F_\parallel$, namely

$$F_\parallel^I((x_i)_{i \in I}) = \min_{i \in I} |x_i| 1_{\{x_i \geqslant 0, \forall i \in I \text{ or } x_i \leqslant 0, \forall i \in I\}} \prod_{i \in I} \operatorname{sgn} x_i. \tag{4.7}$$

Therefore, the one-dimensional margins satisfy $F^{\{i\}}(u) = u$.

$\Rightarrow$: Let $x \in (0, \infty)^d$. Clearly, $U(x) \leqslant U_k(x_k)$ for any k. On the other hand, since S is an ordered set, we have

$$\{y \in \mathbb{R}^d : x_k \leqslant y_k\} \cap S = \{y \in \mathbb{R}^d : x \geqslant y\} \cap S$$

for some k. Indeed, suppose that this is not so. Then there exist points $z^1, \ldots, z^d \in S$ and indices $j_1, \ldots, j_d$ such that $z_k^k \geqslant x_k$ and $z_{j_k}^k < x_{j_k}$ for $k = 1, \ldots, d$. Choose the greatest element among $z^1, \ldots, z^d$ (this is possible because they all belong to an ordered set) and call it z^k. Then $z_{j_k}^k < x_{j_k}$. However, by construction of $z^1, \ldots, z^d$ we also have $z_{j_k}^{j_k} \geqslant x_{j_k}$, which is a contradiction to the fact that z^k is the greatest element. Therefore,

$$U(x) = \min(U_1(x_1), \ldots, U_d(x_d)).$$

Similarly, it can be shown that for every $x \in (-\infty, 0)^d$,

$$U(x) = (-1)^d \min(|U_1(x_1)|, \ldots, |U_d(x_d)|).$$

Since $U(x) = 0$ for any $x \notin K$, we have shown that

$$U(x) = F_\parallel(U_1(x_1), \ldots, U_d(x_d))$$

for any $x \in (\mathbb{R}\backslash\{0\})^d$. Since the marginal Lévy measures of X are also supported by non-decreasing sets and the margins of $F_\|$ have the same form as $F_\|$, we have

$$U^I((x_i)_{i\in I}) = F_\|^I((U_i(x_i))_{i\in I}) \tag{4.8}$$

for any $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$.

$\Leftarrow$: Let $S := \operatorname{supp} \nu$. Let us first show that $S \subset K$. Suppose that this is not so. Then there exists $x \in S$ such that for some m and n, $x_m < 0$ and $x_n > 0$ and for every neighbourhood N of x, $\nu(N) > 0$. This implies that $U^{\{m,n\}}(x_m/2, x_n/2) > 0$, which contradicts Eq. (4.8).

Suppose now that S is not an ordered set. Then there exist two points $u, v \in S$ such that $u_m > v_m$ and $u_n < v_n$ for some m and n. Moreover, we can have either $u_i \geqslant 0$ and $v_i \geqslant 0$ for all i or $u_i \leqslant 0$ and $v_i \leqslant 0$ for all i. Suppose that $u_i \geqslant 0$ and $v_i \geqslant 0$, the other case being analogous. Let $x = \frac{u+v}{2}$. Since $u, v \in S$, we have $\nu(\{z \in \mathbb{R}^d : z_m < x_m, z_n \geqslant x_n\}) > 0$ and $\nu(\{z \in \mathbb{R}^d : z_m \geqslant x_m, z_n < x_n\}) > 0$. However

$$\begin{aligned}\nu(\{z \in \mathbb{R}^d : z_m < x_m, z_n \geqslant x_n\}) &= U_n(x_n) - U^{\{m,n\}}(x_m, x_n)\\ &= U_n(x_n) - \min(U_m(x_m), U_n(x_n))\end{aligned}$$

and

$$\nu(\{z \in \mathbb{R}^d : z_m \geqslant x_m, z_n < x_n\}) = U_m(x_m) - \min(U_m(x_m), U_n(x_n)),$$

which is a contradiction because these expressions cannot be simultaneously positive.

For the last assertion, we assume that the tail integrals U_i of X^i are continuous and satisfy $\lim_{x\to 0} U_i(x) = \infty$, $i = 1, \ldots, d$. The uniqueness of the copula readily follows from Theorem 3.6. It suffices to show that $\nu(S_n) = 0$ for any n in decomposition (4.4). If $\xi(n) \neq 0$, then

$$\nu(S_n) = \lim_{\varepsilon\downarrow 0} (U_{k(n)}(\xi(n) - \varepsilon) - U_{k(n)}(\xi(n))) = 0,$$

because $U_{k(n)}$ is continuous. Suppose now that $\xi(n) = 0$. Since S_n does not reduce to a single point, we must have either $x_m > 0$ or $x_m < 0$ for some $x \in S_n$ and some m. Suppose that $x_m > 0$, the other case being analogous. Since S is ordered, we have

$$\nu(\{x \in \mathbb{R}^d : x_{k(n)} \geqslant \varepsilon\} \cap S) \leqslant \nu(\{\xi \in \mathbb{R}^d : \xi_m \geqslant x_m\} \cap S) < \infty$$

uniformly in $\varepsilon > 0$. This implies $\lim_{x\downarrow 0} U_{k(n)}(x) < \infty$ in contradiction to $\lim_{x\to 0} U_{k(n)}(x) = \infty$. Hence, $\xi(n) > 0$ for any n. Therefore, $\nu(\mathbb{R}^d \setminus S^*) = 0$ and the proof is completed. $\square$

Lévy copulas provide a simple characterization of possible dependence patterns of multivariate stable Lévy motions.

Theorem 4.6. *Let X be a $\mathbb{R}^d$-valued Lévy process and let $\alpha \in (0, 2)$. X is α-stable if and only if its components $X^1, \ldots, X^d$ are α-stable and if it has a Lévy copula F that is a homogeneous function of order* 1, *i.e.*

$$F(ru_1, \ldots, ru_d) = rF(u_1, \ldots, u_d) \tag{4.9}$$

for any $r > 0, u_1, \ldots, u_d \in \mathbb{R}^d$.

Proof. $\Rightarrow$: The Lévy measure of a one-dimensional α-stable distribution has a density given by

$$x \mapsto \frac{A}{x^{1+\alpha}} 1_{\{x>0\}} + \frac{B}{|x|^{1+\alpha}} 1_{\{x \leqslant 0\}}$$

for some $A \geqslant 0$ and $B \geqslant 0$ (Theorem 14.3 in [19]). Consequently, three situations are possible for any $i = 1, \dots, d$, namely $\operatorname{Ran} U_i = (-\infty, 0]$ (only negative jumps), $\operatorname{Ran} U_i = [0, \infty)$ (only positive jumps), or $\operatorname{Ran} U_i = \mathbb{R} \setminus \{0\}$ (jumps of both signs). We exclude the trivial case of a component having no jumps at all. Let $I_1 = \{i : \operatorname{Ran} U_i = (-\infty, 0]\}$ and $I_2 = \{i : \operatorname{Ran} U_i = [0, \infty)\}$. For any i, let $\overline{X}^i$ be a copy of X^i, independent of X and of $\overline{X}^k$ for $k \neq i$. Define a $\mathbb{R}^d$-valued Lévy process $\widetilde{X}$ by

$$\widetilde{X}^i = \begin{cases} X^i, & i \notin I_1 \cup I_2, \\ X^i - \overline{X}^i, & i \in I_1 \cup I_2. \end{cases}$$

Denote by $\widetilde{\nu}$ the Lévy measure of $\widetilde{X}$, by $\widetilde{U}$ its tail integral, and by $\widetilde{F}$ its Lévy copula. From the construction of $\widetilde{X}$ it follows that

$$U^I((x_i)_{i\in I}) = \widetilde{U}^I((x_i)_{i\in I}) \tag{4.10}$$

for any $I \subset \{1, \dots, d\}$ with card $I \geqslant 2$ and any $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. The process $\widetilde{X}$ is clearly α-stable and each component of this process has jumps of both signs (i.e. $\operatorname{Ran} \widetilde{U}_i = \mathbb{R} \setminus \{0\}$). By Theorem 14.3 in [19] we have

$$\widetilde{\nu}(B) = r^\alpha \widetilde{\nu}(rB) \tag{4.11}$$

for any $B \in \mathcal{B}(\mathbb{R}^d)$ and for any $r > 0$. Therefore,

$$\widetilde{U}^I((x_i)_{i\in I}) = r^\alpha \widetilde{U}^I((rx_i)_{i\in I}) \tag{4.12}$$

for any nonempty $I \subset \{1, \dots, d\}$ and any $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. By Theorem 3.6 this implies

$$\widetilde{F}^I((u_i)_{i\in I}) = r^{-1} \widetilde{F}^I((ru_i)_{i\in I})$$

for any $(u_1, \dots, u_d) \in (\mathbb{R} \setminus \{0\})^d$. Therefore (4.9) holds for $\widetilde{F}$. It remains to prove that $\widetilde{F}$ is also a Lévy copula of X. Indeed, let $I \subset \{1, \dots, d\}$ nonempty and $(x_i)_{i\in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$. Two situations are possible:

- $\widetilde{U}_i(x_i) = U(x_i)$ for every $i \in I$. Then (4.10) implies that Equation (3.2) holds with F replaced by $\widetilde{F}$.
- $U_k(x_k) = 0$ for some $k \in I$. Then $\widetilde{F}^I((U_i(x_i))_{i\in I}) = 0$, but on the other hand $|U^I((x_i)_{i\in I})| \leqslant |U_k(x_k)| = 0$ and (3.2) holds again with F replaced by $\widetilde{F}$.

$\Leftarrow$: Since X has α-stable margins and a homogeneous Lévy copula, its marginal tail integrals satisfy (4.12). From the proof of Lemma 3.5 it follows that the Lévy measure of every set of the form $(a, b]$ can be expressed as a limit of linear combinations of tail integrals. Therefore, the Lévy measure of X has the property (4.11). We conclude from Theorem 14.3 in [19] that X is α-stable. $\square$

5. Lévy copulas as limits of ordinary copulas

In this section, we explore the relation between the Lévy copula F of a Lévy process X and the (ordinary) copula C_t of its distribution at a given time t. It turns out that in all points where the Lévy copula is unique (cf. Theorem 3.6), it is completely determined by the limiting behavior of C_t as $t \to 0$. Moreover, we shall see that it is only the behavior of C_t in the corners of its domain of definition (which is $[0, 1]^d$) that matters.

A *copula* is a function $C : [0, 1]^d \to [0, 1]$ such that C is d-increasing, $C(u_1, \dots, u_d) = 0$ if $u_i = 0$ for some i, and $C(u_1, \dots, u_d) = u_k$ if $u_i = 1$ for all $i \neq k$. Let $X = (X^1, \dots, X^d)$ be a $\mathbb{R}^d$-valued random variable with distribution function

$$H(x_1, \dots, x_d) := P[X^1 \leqslant x_1, \dots, X^d \leqslant x_d]$$

and marginal distribution functions $H_i(x) := P[X^i \leqslant x]$. The *copula of X* or the *copula of H* is any copula C such that

$$C(H_1(x_1), \dots, H_d(x_d)) = H(x_1, \dots, x_d)$$

for all $(x_1, \dots, x_d) \in \mathbb{R}^d$. The *survival function* of X is defined by

$$\overline{H}(x_1, \dots, x_d) = P[X^1 > x_1, \dots, X^d > x_d],$$

and the *survival copula* $\overline{C}$ of X is a copula that relates the survival function of X to its marginal survival functions

$$\overline{C}(\overline{H}_1(x_1), \dots, \overline{H}_d(x_d)) = \overline{H}(x_1, \dots, x_d)$$

for all $(x_1, \dots, x_d) \in \mathbb{R}^d$. Using the continuity of copulas [14, Theorem 2.10.7] it is not hard to verify that any copula of X is also a survival copula of $-X$ and vice versa.

Theorem 5.1. *Let X be a $\mathbb{R}^d$-valued Lévy process with marginal tail integrals $U_1, \dots, U_d$, and denote by F its Lévy copula in the sense of Theorem* 3.6. *Denote by $C_t^{(\alpha_1,\dots,\alpha_d)} : [0, 1]^d \to [0, 1]$ an (ordinary) copula of $(-\alpha_1 X_t^1, \dots, -\alpha_d X_t^d)$ (or, equivalently, a survival copula of $(\alpha_1 X_t^1, \dots, \alpha_d X_t^d)$) for $t \in (0, \infty)$, $\alpha \in \{-1, 1\}^d$. Then*

$$F(u_1, \dots, u_d) = \lim_{t\to 0} \frac{1}{t} C_t^{(\operatorname{sgn} u_1,\dots,\operatorname{sgn} u_d)}(t|u_1|, \dots, t|u_d|) \prod_{i=1}^{d} \operatorname{sgn} u_i. \tag{5.1}$$

for any $(u_1, \dots, u_d) \in \prod_{i=1}^d \overline{\operatorname{Ran} U_i}$.

Remark. Note that although $C_t^{(\operatorname{sgn} u_1,\dots,\operatorname{sgn} u_d)}(t|u_1|, \dots, t|u_d|)$ is not defined for $t|u_i| > 1$, the limit for $t \to 0$ still makes sense.

Proof. *Step* 1: By Theorem 2.10.7 in [14] and Lemma 3.2 above, copulas and Lévy copulas are Lipschitz with a common Lipschitz constant. Consequently, it suffices to prove the assertion for $(u_1, \dots, u_d) \in \prod_{i=1}^d \operatorname{Ran} U_i$.

Step 2: It suffices to prove the assertion for nonnegative $u_1, \dots, u_d$. Indeed, let $(u_1, \dots, u_d) \in \prod_{i=1}^d \operatorname{Ran} U_i$ and $\alpha_i := \operatorname{sgn} u_i$, $i = 1, \dots, d$. Let $\widetilde{X} := (\alpha_1 X^1, \dots, \alpha_d X^d)$ and denote by $\widetilde{F}$ a

Lévy copula of $\widetilde{X}$ and by $\widetilde{U}$ and $\widetilde{U}_i$ its tail integrals. Theorem 3.6 and Lemma 3.2 imply that

$$\begin{aligned}
U(x_1,\ldots,x_d) &= \lim_{\xi_i \downarrow x_i, i=1,\ldots,d} \widetilde{U}(\alpha_1\xi_1,\ldots,\alpha_d\xi_d)\prod_{i=1}^d \alpha_i \\
&= \lim_{\xi_i \downarrow x_i, i=1,\ldots,d} \widetilde{F}\left(\widetilde{U}_1(\alpha_1\xi_1),\ldots,\widetilde{U}_d(\alpha_d\xi_d)\right)\prod_{i=1}^d \alpha_i \\
&= \widetilde{F}\left(\lim_{\xi_1\downarrow x_1}\widetilde{U}_1(\alpha_1\xi_1),\ldots,\lim_{\xi_d\downarrow x_d}\widetilde{U}_1(\alpha_1\xi_1)\right)\prod_{i=1}^d \alpha_i \\
&= \widetilde{F}(\alpha_1 U_1(x_1),\ldots,\alpha_d U_d(x_d))\prod_{i=1}^d \alpha_i.
\end{aligned}$$

for any $x_1,\ldots,x_d \in \mathbb{R}\setminus\{0\}$. Therefore,

$$F(u_1,\ldots,u_d) = \widetilde{F}(|u_1|,\ldots,|u_d|)\prod_{i=1}^d \alpha_i = \lim_{t\to 0}\frac{1}{t}C_t^{(\alpha_1,\ldots,\alpha_d)}(t|u_1|,\ldots,t|u_d|)\prod_{i=1}^d \alpha_i,$$

if the assertion holds for the Lévy process $\widetilde{X}$ and $(|u_1|,\ldots,|u_d|) \in \prod_{i=1}^d \overline{\operatorname{Ran}\widetilde{U}_i}$.

Step 3: Let $u_i = U_i(x_i)$ with $x_i > 0$ for $i = 1,\ldots,d$. Choose $x_0 > 0$ small enough such that $\nu(\mathbb{R}^d \setminus [-x_0,x_0]^d) > 0$ and let

$$A := \left\{y \in \mathbb{R}^d : |y_i| \geqslant \min(x_0, x_i/2) \text{ for some } i = 1,\ldots,d\right\}.$$

Choose a continuous mapping $g : \mathbb{R}^d \to [0,1]$ such that $g(y) = 1$ if $y \in A$ and $g(y) = 0$ in a neighbourhood of 0. For $t \in (0,\infty)$ define a probability measure P_t^g on $\mathbb{R}^d$ via

$$P_t^g(B) := \frac{E[1_B(X_t)g(X_t)]}{E[g(X_t)]}.$$

Observe that P_t^g is well defined if $\nu \neq 0$, because in this case the support of P_{X_t} is unbounded for any $t > 0$ [19, Th. 24.3]. Let $c_t := E[g(X_t)]$. Then $P_t^g(B) = \frac{1}{c_t}P_{X_t}(B)$ for any $B \in \mathcal{B}(\mathbb{R}^d)$ with $B \subset A$. This implies

$$\overline{C}_t(\overline{H}_{t,1}(y_1),\ldots,\overline{H}_{t,d}(y_d)) = \frac{1}{c_t}C_t^{(1,\ldots,1)}\left(c_t\overline{H}_{t,1}(y_1),\ldots,c_t\overline{H}_{t,d}(y_d)\right) \tag{5.2}$$

for any $y_i \geqslant x_i/2$, where $\overline{C}_t$ denotes a survival copula of P_t^g and $\overline{H}_{t,i}, i = 1,\ldots,d$ are the survival functions of the marginals of P_t^g.

Step 4: Define a probability measure Q^g on $\mathbb{R}^d$ by

$$Q^g(B) := \frac{\int g 1_B \, d\nu}{\int g \, d\nu}.$$

Denote by $\overline{C}$ a survival copula of Q^g, by $\overline{H}_i, i = 1,\ldots,d$ the marginal survival functions of Q^g, and let $c := \int g d\nu$. Then $\overline{H}_i(x_i) = \frac{u_i}{c}$ and

$$F(u_1,\ldots,u_d) = c\overline{C}\left(\frac{u_1}{c},\ldots,\frac{u_d}{c}\right)$$

follows similarly as (5.2) from the definition of $\overline{C}$ and F. Here, $u_1,\ldots,u_d$ denote the numbers from Step 3.

Step 5: By [19, Corollary 8.9] we have $\frac{1}{t}\int f\,dP_{X_t} \stackrel{t\to 0}{\longrightarrow} \int f\,d\nu$ for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ which vanishes in a neighbourhood of 0. This implies $\frac{1}{t}\int fg\,dP_{X_t} \stackrel{t\to 0}{\longrightarrow} \int fg\,d\nu$ and hence

$$\frac{c_t}{ct}\int f\,dP_t^g \stackrel{t\to 0}{\longrightarrow} \int f\,dQ^g$$

for any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$. Therefore,

$$\frac{c_t}{ct}P_t^g \to Q^g$$

weakly for $t \to 0$. In particular $\frac{c_t}{ct} \to 1$, which implies that $P_t^g \to Q^g$ as well. Therefore, $\overline{C}_t(\frac{u_1}{c},\dots,\frac{u_d}{c}) \to \overline{C}(\frac{u_1}{c},\dots,\frac{u_d}{c})$ for $t \to 0$ because weak convergence implies pointwise convergence of the copulas [10, Theorem 2.1]. Since $P_t^g \to Q^g$, we have that $\overline{H}_{t,i}$ converges to $\overline{H}_i$ on a dense set. Hence, there exist $y_{t,i} \to x_i$ with $\overline{H}_{t,i}(y_{t,i}) \to \overline{H}_i(x_i) = \frac{u_i}{c}$ for $t \to 0$.

Step 6: Together, it follows that

$$\begin{aligned} F(u_1,\dots,u_d) &= c\overline{C}\left(\frac{u_1}{c},\dots,\frac{u_d}{c}\right)\\ &= \lim_{t\to 0} c\overline{C}_t\left(\frac{u_1}{c},\dots,\frac{u_d}{c}\right)\\ &= \lim_{t\to 0} c\overline{C}_t\left(\overline{H}_{t,1}(y_{t,1}),\dots,\overline{H}_{t,d}(y_{t,d})\right)\\ &= \lim_{t\to 0} \frac{c}{c_t}C_t^{(1,\dots,1)}\left(c_t\overline{H}_{t,1}(y_{t,1}),\dots,c_t\overline{H}_{t,d}(y_{t,d})\right)\\ &= \lim_{t\to 0} \frac{1}{t}C_t^{(1,\dots,1)}(tu_1,\dots,tu_d)\,. \end{aligned}$$

In the third and the last equality we used the fact that copulas are Lipschitz with a common Lipschitz constant [14, Theorem 2.10.7]. □

Remark. The tail copulas F^I of F are obtained accordingly by considering the copulas of X^I instead of X.

Example 5.2. Let X be a $\mathbb{R}^d$-valued stable Lévy process with Lévy copula F. For any $a > 0$ there exist $b > 0$ and $c \in \mathbb{R}^d$ such that $X_{at} \stackrel{d}{=} bX_t + ct$, $t \in \mathbb{R}_+$. This implies that the copula of X_t is also the copula of X_{at} for all $t \in \mathbb{R}_+$, $a > 0$. By Theorem 5.1, we have

$$F(u_1,\dots,u_d) = \lim_{t\to 0}\frac{1}{t}C^{(\operatorname{sgn} u_1,\dots,\operatorname{sgn} u_d)}(t|u_1|,\dots,t|u_d|)\prod_{i=1}^d \operatorname{sgn} u_i,$$

for any $(u_1,\dots,u_d) \in \prod_{i=1}^d \overline{\operatorname{Ran} U_i}$, where $C^{\alpha_1,\dots,\alpha_d}$ is the survival copula of $(\alpha_1 X_t^1,\dots,\alpha_d X_t^d)$ for any fixed $t > 0$.

Remark. Theorem 5.1 shows that the Lévy copula is determined by the ordinary copulas C_t for small t. By contrast, it is generally not possible to derive the copulas C_t from the Lévy copula. Indeed, given two multivariate Lévy processes with the same Lévy copula, their copulas C_t for fixed t do not necessarily coincide. The following example shows that this happens even in the special case of the complete dependence Lévy copula (4.3).

Let N^1 and N^2 be independent standard Poisson processes. First, consider the two-dimensional Lévy process with equal components $X = (N^1, N^1)$. Then X has completely dependent jumps and hence the complete dependence Lévy copula $F_\|$ is a possible Lévy copula for X. Moreover, for any t the ordinary complete dependence copula $C_\|(u, v) = \min(u, v)$ is a possible copula for X_t. Now let $Y^1 = 2N^1 + 3N^2$ and $Y^2 = N^1 + 4N^2$. Then $Y = (Y^1, Y^2)$ is also a two-dimensional Lévy process with completely dependent jumps and hence $F_\|$ is a possible Lévy copula for Y as well. However, for any $t > 0$ the copula of Y_t is different from the complete dependence copula $C_\|$: Since

$$\begin{aligned}
P[Y_t^1 \leqslant 3] &= P[N_t^1 = 0; N_t^2 = 0] + P[N_t^1 = 1; N_t^2 = 0] + P[N_t^1 = 0; N_t^2 = 1],\\
P[Y_t^2 \leqslant 2] &= P[N_t^1 = 0; N_t^2 = 0] + P[N_t^1 = 1; N_t^2 = 0] + P[N_t^1 = 2; N_t^2 = 0],\\
P[Y_t^1 \leqslant 3; Y_t^2 \leqslant 2] &= P[N_t^1 = 0; N_t^2 = 0] + P[N_t^1 = 1; N_t^2 = 0],
\end{aligned}$$

we have $P[Y_t^1 \leqslant 3; Y_t^2 \leqslant 2] < P[Y_t^1 \leqslant 3]$ and $P[Y_t^1 \leqslant 3; Y_t^2 \leqslant 2] < P[Y_t^2 \leqslant 2]$. If the complete dependence copula $C_\|$ were a possible copula of Y_t, we would have $P[Y_t^1 \leqslant 3; Y_t^2 \leqslant 2] = \min(P[Y_t^1 \leqslant 3], P[Y_t^2 \leqslant 2])$.

The dependence between the components of a Lévy process X is not entirely characterized by the Lévy copula because X may also have a Brownian motion part B. Because of the scaling property of Brownian motion, the copula of the random vector B_t does not depend on t (cf. Example 5.2). Since B_t has a multivariate normal distribution, this copula C^B is a Gaussian copula. The following theorem shows that it can also be recovered as a limit of the copulas of X_t for $t \to 0$.

Theorem 5.3. *Let X be a $\mathbb{R}^d$-valued Lévy process and denote by C^B the Gaussian copula of the continuous martingale part $B = (B^1, \ldots, B^d)$* of X (*which is possibly 0*). *For $t > 0$ denote by $C_t : [0, 1]^d \to [0, 1]$ the probabilistic copula of* $(-X_t^1, \ldots, -X_t^d)$ (*i.e.*, $C_t = C_t^{(1,\ldots,1)}$ *in the notation of Theorem* 5.1). *Then we have*

$$C^B(u_1, \ldots, u_d) = \lim_{t\to 0} C_t(u_1, \ldots, u_d)$$

for any $(u_1, \ldots, u_d) \in V$, where V denotes the subset of $[0, 1]^d$ *where the Gaussian copula C^B is uniquely defined.*

Remark. If no component of B equals zero, then the margins of B_t are continuous for any fixed t and therefore the Gaussian copula C^B is unique.

Proof. Choose a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \downarrow 0$ for $n \to \infty$. Write $X = B + L$, where B is a Brownian motion and L is a Lévy process without continuous martingale part. Denote by $(0, \nu t, \gamma t)$ the Lévy–Khintchine triplet of L_t relative to the truncation function $x \mapsto x1_{\{|x|\leqslant 1\}}$ (cf. e.g. [19], Definition 8.2). Straightforward calculations yield that the corresponding triplet of $t_n^{-1/2} L_{t_n}$ equals

$$\left(0, \nu^{(n)} t_n, \left(\gamma - \int x 1_{\{\sqrt{t_n}<|x|\leqslant 1\}}\nu(dx)\right)\sqrt{t_n}\right),$$

where $\nu^{(n)}(A) := \int 1_A(t_n^{-1/2}x)\nu(dx)$, $A \in \mathcal{B}(\mathbb{R}^d)$. By dominated convergence we have that

$$\left|\left(\gamma - \int x 1_{\{\sqrt{t_n}<|x|\leqslant 1\}}\nu(dx)\right)\sqrt{t_n}\right| \leqslant |\gamma|\sqrt{t_n} + \int_{\{|x|\leqslant 1\}}\left(|x|^2 \wedge |x|\sqrt{t_n}\right)\nu(dx) \stackrel{n\to\infty}{\longrightarrow} 0$$

and similarly

$$\int |x|^2 1_{\{|x|\leqslant 1\}}\nu^{(n)}(dx)t_n \stackrel{n\to\infty}{\longrightarrow} 0,$$

$$\int g(x)\nu^{(n)}(dx)t_n \stackrel{n\to\infty}{\longrightarrow} 0$$

for any bounded function $g : \mathbb{R}^d \to \mathbb{R}$ which vanishes in a neighbourhood of 0. From [5, Theorem VII.2.9], it follows that $t_n^{-1/2}L_{t_n}$ converges weakly to 0 for $n \to \infty$. Moreover, we have $t^{-1/2}B_t \stackrel{d}{=} B_1$ for any $t \in \mathbb{R}_+$. Consequently, we have $t_n^{-1/2}X_{t_n} \to B_1$ weakly for $n \to \infty$. From [10, Theorem 2.1], it follows that the corresponding sequence of copulas converges as well. But note that the copula of $t_n^{-1/2}X_{t_n}$ coincides for any $n \in \mathbb{N}$ with the copula of X_{t_n}. Therefore, the copula of X_{t_n} converges to the copula of B_1 on the set where the latter is uniquely defined. □

Let us finally try to explain the intuition behind the two preceding theorems. For small times t, the Lévy process X_t is very likely to be found close to the origin. With small probability, on the other hand, it is carried to macroscopic values, typically by a single large jump. Consequently, the law of X_t away from the origin resembles the Lévy measure ν because the latter is closely related to the distribution of single jumps [19, Corollary 8.9]. But for small t, jumps above a certain size become less and less likely. On the level of the copula of X_t, these extreme jump events of small probability are reflected in the corners, which may indicate that Theorem 5.1 makes sense.

Since jumps of a size above a certain level practically do not happen in very small periods of time, the unrescaled copula of X_t resembles the Gaussian copula of the Brownian motion part in the limit. It may seem somewhat surprising that this extends also to the case that the Lévy measure is infinite near the origin. In this case the jump part may be of unbounded variation and therefore not entirely different from Brownian motion.

6. Parametric families of Lévy copulas

The following result is analogous to the Archimedean copula construction (cf. e.g. [14]). It allows to obtain parametric families of Lévy copulas in arbitrary dimension, where the number of parameters does not depend on the dimension.

Theorem 6.1. *Let* $\varphi : [-1, 1] \to [-\infty, \infty]$ *be a strictly increasing continuous function with* $\varphi(1) = \infty$, $\varphi(0) = 0$, *and* $\varphi(-1) = -\infty$, *having derivatives of orders up to d on* $(-1, 0)$ *and* $(0, 1)$, *and satisfying*

$$\frac{d^d\varphi(e^x)}{dx^d} \geqslant 0, \quad \frac{d^d\varphi(-e^x)}{dx^d} \leqslant 0, \quad x \in (-\infty, 0). \tag{6.3}$$

Let

$$\tilde{\varphi}(u) := 2^{d-2}(\varphi(u) - \varphi(-u))$$

for $u \in [-1, 1]$. *Then*

$$F(u_1, \dots, u_d) := \varphi\left(\prod_{i=1}^{d} \widetilde{\varphi}^{-1}(u_i)\right)$$

defines a Lévy copula.

Proof. Firstly, note that $\widetilde{\varphi}$ is a strictly increasing continuous function from $[-1, 1]$ to $[-\infty, \infty]$, satisfying $\widetilde{\varphi}(1) = \infty$ and $\widetilde{\varphi}(-1) = -\infty$, which means that $\widetilde{\varphi}^{-1}$ exists for all $u \in \overline{\mathbb{R}}$ and F is well defined. Properties 1 and 2 of Definition 3.1 are clearly satisfied. For $k = 1, \dots, d$ and $u_k \in \mathbb{R}$ we have

$$\begin{aligned} F^{\{k\}}(u_k) &= \lim_{c\to\infty} \sum_{(u_i)_{i\neq k}\in\{-c,\infty\}^{d-1}} F(u_1, \dots, u_d) \prod_{i\neq k} \operatorname{sgn} u_i \\ &= \sum_{(u_i)_{i\neq k}\in\{-\infty,\infty\}^{d-1}} \varphi\left(\widetilde{\varphi}^{-1}(u_k) \prod_{i\neq k} \operatorname{sgn} u_i\right) \prod_{i\neq k} \operatorname{sgn} u_i \\ &= \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^i \varphi\left(\widetilde{\varphi}^{-1}(u_k)(-1)^i\right) \\ &= 2^{d-2}\left(\varphi(\widetilde{\varphi}^{-1}(u_k)) - \varphi(-\widetilde{\varphi}^{-1}(u_k))\right) = u_k, \end{aligned}$$

which proves property (4). It remains to show that F is d-increasing. Since $\widetilde{\varphi}^{-1}$ is increasing, it suffices to show that $(u_1, \dots, u_d) \mapsto \varphi(\prod_{i=1}^d u_i)$ is d-increasing on $(-1, 1]^d$. Since $\varphi(\prod_{i=1}^d u_i) = \varphi(\prod_{i=1}^d |u_i| \prod_{i=1}^d \operatorname{sgn} u_i)$, it suffices to prove that both $(u_1, \dots, u_d) \mapsto \varphi(\prod_{i=1}^d u_i)$ and $(u_1, \dots, u_d) \mapsto -\varphi(-\prod_{i=1}^d u_i)$ are d-increasing on $[0, 1]^d$ or, equivalently, on $(0, 1)^d$ (since φ is continuous). The first condition of (6.3) implies that

$$\frac{\partial^d \psi(z_1, \dots, z_d)}{\partial z_1 \dots \partial z_d} \geqslant 0$$

on $(-\infty, 0)^d$ for $\psi(z_1, \dots, z_d) := \varphi(e^{z_1+\dots+z_d})$. From Definition 2.1 it follows easily that

$$V_\psi(B) = \int_B \frac{\partial^d \psi(z_1, \dots, z_d)}{\partial z_1 \dots \partial z_d} dz_1 \dots dz_d.$$

Therefore, ψ is increasing on $(-\infty, 0)^d$, which implies that $(u_1, \dots, u_d) \mapsto \varphi(\prod_{i=1}^d u_i)$ is d-increasing on $(0, 1)^d$. The second condition of (6.3) entails similarly that $(u_1, \dots, u_d) \mapsto -\varphi(-\prod_{i=1}^d u_i)$ is d-increasing on $(0, 1)^d$ as well. □

Remark. Condition (6.3) is satisfied in particular if for any $k = 1, \dots, d$,

$$\frac{d^k \varphi(u)}{du^k} \geqslant 0, \quad u \in (0, 1) \quad \text{and} \quad (-1)^k \frac{d^k \varphi(u)}{du^k} \leqslant 0, \quad u \in (-1, 0).$$

Example 6.2. Let

$$\varphi(x) := \eta(-\log|x|)^{-1/\vartheta} 1_{\{x>0\}} - (1-\eta)(-\log|x|)^{-1/\vartheta} 1_{\{x<0\}}$$

with $\vartheta > 0$ and $\eta \in (0, 1)$. Then

$$\widetilde{\varphi}(x) = 2^{d-2}(-\log|x|)^{-1/\vartheta}\operatorname{sgn} x,$$
$$\widetilde{\varphi}^{-1}(u) = \exp(-|2^{2-d}u|^{-\vartheta})\operatorname{sgn} u,$$

and therefore

$$F(u_1, \ldots, u_d) = 2^{2-d}\left(\sum_{i=1}^{d}|u_i|^{-\vartheta}\right)^{-1/\vartheta}\left(\eta 1_{\{u_1\cdots u_d \geqslant 0\}} - (1-\eta)1_{\{u_1\cdots u_d < 0\}}\right) \tag{6.4}$$

defines a two-parameter family of Lévy copulas. It resembles the Clayton family of ordinary copulas. F is in fact a Lévy copula for any $\vartheta > 0$ and any $\eta \in [0, 1]$. The role of the parameters is easiest to analyze in the case $d = 2$, when (6.4) becomes

$$F(u, v) = (|u|^{-\vartheta} + |v|^{-\vartheta})^{-1/\vartheta}\left(\eta 1_{\{uv \geqslant 0\}} - (1-\eta)1_{\{uv<0\}}\right). \tag{6.5}$$

From this equation it is readily seen that the parameter η determines the dependence of the *sign* of jumps: when $\eta = 1$, the two components always jump in the same direction, and when $\eta = 0$, positive jumps in one component are accompanied by negative jumps in the other and vice versa. For intermediate values of η, positive jumps in one component can correspond to both positive and negative jumps in the other component. The parameter ϑ is responsible for the dependence of absolute values of jumps in different components. In particular, F converges to the independence Lévy copula (4.1) if $\eta = 1$ and $\vartheta \to 0$. Similarly, F tends to the Lévy copula of complete dependence (4.3) for $\eta = 1$ and $\vartheta \to \infty$.

References

[1] O. Barndorff-Nielsen, Processes of normal inverse Gaussian type, Finance & Stochastics 2 (1998) 41–68.

[2] O.E. Barndorff-Nielsen, J. Pedersen, Ken-Iti Sato, Multivariate subordination, self-decomposability and stability, Adv. in Appl. Probab. 33 (1) (2001) 160–187.

[3] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC, London, Boca Raton, FL, 2004.

[4] E. Eberlein, Application of generalized hyperbolic Lévy motions to finance, in: O. Barndorff-Nielsen, T. Mikosch, S. Resnick (Eds.), Lévy Processes, Birkhäuser, Basel, 2001, pp. 319–337.

[5] J. Jacod, A. Shiryaev, Limit Theorems for Stochastic Processes, second ed., Springer, Berlin, 2003.

[6] H. Joe, Multivariate Models and Dependence Concepts, Chapman & Hall, London, 1997.

[7] J. Kingman, S. Taylor, Introduction to Measure and Probability, Cambridge University Press, Cambridge, 1966.

[8] S. Kou, A jump-diffusion model for option pricing, Manage. Sci. 48 (2002) 1086–1101.

[9] P. Lévy, Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, (2^e edition, 1954, 1^e ed., 1937).

[10] A. Lindner, A. Szimayer, A limit theorem for copulas, technical report, Technical University of Munich (cf. http://www-4.ma.tum.de/m4/pers/lindner), 2004.

[11] D. Madan, P. Carr, E. Chang, The variance gamma process and option pricing, European Finan. Rev. 2 (1998) 79–105.

[12] M. Maejima, J. Rosiński, The class of type G distributions on $\mathbb{R}^d$ and related subclasses of infinitely divisible distributions, Demonstratio Math. 34 (2) (2001) 251–266.

[13] M. Maejima, J. Rosiński, Type G distributions on $\mathbb{R}^d$, J. Theoret. Probab. 15 (2) (2002) 323–341.

[14] R. Nelsen, An Introduction to Copulas, vol. 139, Lecture Notes in Statistics, Springer, New York, 1999.

[15] K. Prause, The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures, Dissertation Universität Freiburg i. Br., 1999.

[16] J. Rosiński, Tempering stable processes, preprint (cf. http://www.math.utk.edu/~rosinski/manuscripts.html), 2004.

[17] T. Rydberg, The normal inverse Gaussian Lévy process: simulation and approximation, Communications in Statistics. Stochastic Models 13 (1997) 887–910.

[18] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes, Chapman & Hall, New York, 1994.

[19] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.

[20] B. Schweizer, Thirty years of copulas, in: G. Dall'Aglio, S. Kotz, G. Salinetti (Eds.), Advances in probability distributions with given marginals, Kluwer, Dordrecht, 1991.
[21] P. Tankov, Dependence structure of spectrally positive multidimensional Lévy processes, Unpublished manuscript (cf. http://www.math.jussieu.fr/∼tankov), 2003.
[22] P. Tankov, Lévy processes in finance: inverse problems and dependence modelling, PhD thesis, Ecole Polytechnique, France, 2004.